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Gauge Invariance of the Euler–Lagrange Expressions

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We prove, for a Lagrangian density $L(g_{ij}; A_i^\alpha; A_{i,j}^\alpha)$, that the gauge invariance of the Euler–Lagrange expressions $E_\alpha^i(L)$ implies the existence of a gauge-invariant scalar density L_1 such that $E_\alpha^i(L) = E_\alpha^i(L_1)$. We then prove the uniqueness of the Yang–Mills field equations.

1. INTRODUCTION

The Yang–Mills equations for a gauge field

$$B_{\alpha\beta} F^{\beta ij}|_j \tag{1}$$

where

$$F_{ij}^\alpha = A_{j,i}^\alpha - A_{i,j}^\alpha + C_{\beta\gamma}^\alpha A_i^\beta A_j^\gamma \tag{2}$$

A_i^α are the gauge potentials of a connection in a principal fiber bundle, and $C_{\beta\gamma}^\alpha$ are the structure constants [see Kobayashi and Nomizu (1963) and Noriega and Schifini (1985) for definitions and notations; see also Bleecker (1981)], can be obtained through the use of a variational principle as follows. For

$$L = L(g_{ij}; A_i^\alpha; A_{i,j}^\alpha) \tag{3}$$

the Euler–Lagrange expressions are

$$E^i(L) = \frac{\partial_L}{\partial A_i^\alpha} - \frac{\partial}{\partial x^j}\left(\frac{\partial L}{\partial A_{i,j}^\alpha}\right) \tag{4}$$

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If we consider

$$L_0 = B_{\alpha\beta} F^{\alpha ij} F^{\beta}_{ij} \tag{5}$$

then $E^i_\alpha(L)=0$ is equivalent to (1).

The equations (1) are gauge invariant, i.e., they do not change for another choice of gauge because of the transformation law

$$F'^{\alpha ij}|_j = (\mathrm{Ad}^\alpha_\beta \circ \psi^{-1}) F^{\beta ij}|_j$$

for a change of gauge ψ. Also, L_0 is gauge invariant, but while the gauge invariance of the field equations is mandatory, it is not so for the Lagrangian, because it usually has no physical meaning.

In this paper we study how the gauge invariance of the Euler–Lagrange expressions $E^i_\alpha(L)$ affects the possible gauge invariance of the Lagrangian L. In particular, we prove that, under the hypothesis of $E^i_\alpha(L)$ being gauge invariant for some Lagrangian density L of the type (3), there exists a gauge-invariant Lagrangian density L_1 such that $E^i_\alpha(L)=E^i_\alpha(L_1)$. This restricts severely the possible field equations, and we will see that the complete set of field equations leads to Yang–Mills equations, thus proving its uniqueness.

In a previous paper (López *et al.*, 1989) this same result was obtained under the additional hypothesis of the gauge invariance of the contribution to the energy-momentum tensor given by L, i.e., $\partial L/\partial g_{ij}$, in the context of a minimal coupling with gravitation. Since the difficulties of making general relativity and quantum theory compatible are well known, it seems desirable to obtain the above-mentioned uniqueness of the Yang–Mills equations without any reference to gravitation, and this is what we achieve in this paper.

2. THE SEARCH FOR AN EQUIVALENT GAUGE-INVARIANT LAGRANGIAN

The Euler–Lagrange expressions, written out in full, are

$$E^i_\alpha(L) = L^i_\alpha - L^{i,j;h}_{\alpha\ \ \beta} A^\beta_{h,j} - L^{i,j;h,k}_{\alpha\ \ \ \beta} A^\beta_{h,kj} - L^{i,j;hk}_\alpha g_{hk,j} \tag{6}$$

where a comma denotes partial differentiation, L^i_α denotes $\partial L/\partial A^\alpha_i$, $L^{i,j}_\alpha = \partial L/\partial A^\alpha_{i,j}$, and $L^{hk} = \partial L/\partial g_{hk}$.

Differentiating (6) with respect to $g_{hk,j}$, we obtain that $L^{i,j;hk}_\alpha$ is a gauge-invariant tensorial density. By the replacement theorem (Horndeski, 1981)

$$L^{i,j;hk}_\alpha(g_{rs};A^\beta_r;A^\beta_{r,s}) = L^{i,j;hk}_\alpha(g_{rs};0;-\tfrac{1}{2}F^\alpha_{rs}) \tag{7}$$

If $L_\alpha^{i,j;hk}$ denotes the left-hand side of (7) and $L_\alpha^{i,j;hk}(g;0;-\frac{1}{2}F)$ denotes the right-hand side, then we deduce from (7)

$$(L_\alpha^{i,j}-L_\alpha^{i,j}(g;0;-\tfrac{1}{2}F))^{;hk}=0$$

and so $L_\alpha^{i,j}-L_\alpha^{i,j}(g;0;-\frac{1}{2}F)$ is a tensorial density that depends only on A_i^α and $A_{i,j}^\alpha$. Such tensorial densities are known (Calvo *et al.*, to appear), and so

$$L_\alpha^{i,j}-L_\alpha^{i,j}(g;0;-\tfrac{1}{2}F)=d_{\alpha\beta}\varepsilon^{ijhk}A_{h,k}^\beta+d_{\alpha\beta\gamma}\varepsilon^{ijhk}A_h^\beta A_k^\gamma \tag{8}$$

Computing at $A_i^\alpha=0$, $A_{i,j}^\alpha=-\frac{1}{2}F_{ij}^\alpha$, we obtain from (8) that $0=d_{\alpha\beta}\varepsilon^{ijhk}F_{hk}^\beta$.
Then $d_{\alpha\beta}=0$ and we have

$$L_\alpha^{i,j}-L_\alpha^{i,j}(g;0;-\tfrac{1}{2}F)=d_{\alpha\beta\gamma}\varepsilon^{ijhk}A_h^\beta A_k^\gamma \tag{9}$$

We will prove that $L_\alpha^{i,j}$ is skew-symmetric in i, j. First we differentiate (9) with respect to $A_{h,k}^\beta$ and we obtain

$$L_\alpha^{i,j;}{}_\beta^{h,k}-\tfrac{1}{2}[L_\alpha^{i,j;}{}_\beta^{h,k}(g;0;-\tfrac{1}{2}F)-L_\alpha^{i,j;}{}_\beta^{k,h}(g;0;-\tfrac{1}{2}F)]=0$$

Then $L_\alpha^{i,j;}{}_\beta^{h,k}$ is skew-symmetric in h, k. From commutativity of partial derivatives, it follows that it is skew-symmetric in i, j, too. Then

$$(L_\alpha^{i,j}+L_\alpha^{j,i})^{;}{}_\beta^{h,k}=0$$

and so $L_\alpha^{i,j}+L_\alpha^{j,i}$ does not depend of $A_{h,k}^\beta$. Let $H_\alpha^{ij}=L_\alpha^{i,j}+L_\alpha^{j,i}$, i.e., $H_\alpha^{ij}=H_\alpha^{ij}(g;A_r^\beta)$. From (9) we have

$$H_\alpha^{ij}=H_\alpha^{ij}(g_{rs};0)=a_\alpha\sqrt{g}\,g^{ij}$$

where a_α are real numbers. But then

$$L_\alpha^{i,j}=\tfrac{1}{2}(L_\alpha^{i,j}-L_\alpha^{j,i})+\tfrac{1}{2}a_\alpha\sqrt{g}\,g^{ij}$$

and so

$$\begin{aligned}L_\alpha^{i,j;h,k}&=\tfrac{1}{2}(L_\alpha^{i,j;hk}-L_\alpha^{j,i;hk})+\tfrac{1}{4}a_\alpha\sqrt{g}\,g^{ij}g^{hk}\\&\quad-\tfrac{1}{4}a_\alpha\sqrt{g}\,(g^{ih}g^{jk}+g^{ik}g^{jh})\\&=L^{hk}(g;0;-\tfrac{1}{2}F)^{;i,j}_\alpha+\tfrac{1}{4}a_\alpha\sqrt{g}\,(g^{ij}g^{hk}-g^{ih}g^{jk}-g^{ik}g^{jh})\\&=L^{hk}(g;0;-\tfrac{1}{2}F)^{i,j}_\alpha+[a_\beta\sqrt{g}\,A_{r,s}^\beta(g^{ij}g^{hk}-g^{ih}g^{jk}-g^{ik}g^{jh})]^{;i,j}_\alpha\end{aligned}$$

Then

$$\begin{aligned}L^{hk}&=L^{hk}(g;0;-\tfrac{1}{2}F)+a_\beta\sqrt{g}\,A_{r,s}^\beta(g^{rs}g^{hk}-g^{rh}g^{sk}-g^{rk}g^{sh})\\&\quad+T^{hk}(g_{ij};A_i^\alpha)\end{aligned}$$

We deduce that

$$T^{hk}(g_{ij};A_i^\alpha)+a_\beta\sqrt{g}\,A_{r,s}^\beta(g^{rs}g^{hk}-g^{rh}g^{sk}-g^{rk}g^{sh})$$

is a tensorial density. Hence, for a change of coordinates $\bar{x}^i=\bar{x}^i(x^j)$,

$$\begin{aligned}&T^{hk}(B_i^lB_j^mg_{lm};A_i^lA_l^\alpha)+a_\beta\sqrt{g}\,BA_l^rA_m^sA_i^hA_j^k(B_{rs}^pA_p^\beta+B_r^pB_s^qA_{p,q}^\beta)\\&\quad\times(g^{lm}g^{ij}-g^{li}g^{mj}-g^{lj}g^{mi})\\&\quad=BA_u^hA_v^k(T^{uv}(g_{lm};A_i^\alpha)\\&\quad+a_\beta\sqrt{g}\,A_{r,s}^\beta(g^{rs}g^{uv}-g^{ru}g^{sv}-g^{rv}g^{su}))\end{aligned}\tag{10}$$

where $B_i^l=\partial x^l/\partial\bar{x}^i$, $A_m^s=\partial\bar{x}^s/\partial x^m$, $B_{rs}^p=\partial^2x^p/\partial\bar{x}^r\,\partial\bar{x}^s$, and $B=\det(B_j^i)$.

Differentiating (10) with respect to B_{bc}^a and evaluating at $B_b^a=\delta_b^a$, we have

$$a_\beta\sqrt{g}\,A_a^\beta(g^{bc}g^{hk}-g^{bh}g^{ck}-g^{bk}g^{ch})=0$$

Contracting with $g_{bc}g_{hk}$, we find

$$a_\beta\sqrt{g}\,A_a^\beta(16-4-4)=0,\qquad\text{or}\quad a_\beta\sqrt{g}\,A_a^\beta=0\tag{11}$$

and finally, differentiating (11) with respect to A_i^α, we obtain $a_\alpha=0$. Then

$$L_\alpha^{i,j}=-L_\alpha^{j,i}\tag{12}$$

Let

$$P=P(g_{ij};F_{ij}^\alpha)=L(g_{ij};0;-\tfrac{1}{2}F_{ij}^\alpha)$$

Then, by (12)

$$-\tfrac{1}{2}L_\alpha^{i,j}(g;0;-\tfrac{1}{2}F)=P_\alpha^{ij}=\frac{\partial P}{\partial F_{ij}^\alpha}\tag{13}$$

The last identity implies that $P_\alpha^{ij;hk}$ is gauge invariant. Let us consider, for $a\in G$ (the Lie group of the principal fiber bundle)

$$P^{hk}(g_{ij};\mathrm{Ad}_\beta^\alpha(a^{-1})F_{ij}^\beta)-P^{hk}(g_{ij};F_{ij}^\alpha)\tag{14}$$

Differentiating (14) with respect to F_{ij}^β, we obtain

$$P^{hk;ij}_{\alpha}\,\mathrm{Ad}_\beta^\alpha(a^{-1})-P^{hk;ij}_\beta=0$$

because of the gauge invariance of $P_\alpha^{hk;ij}$. Then

$$P^{hk}(g_{ij};\mathrm{Ad}_\beta^\alpha(a^{-1})F_{ij}^\beta)-P^{hk}(g_{ih};F_{ij}^\alpha)=c(a)\sqrt{g}\,g^{hk}\tag{15}$$

Since the right-hand side of (15) does not depend on F_{ij}^α, we evaluate at $F_{ij}^\alpha=0$ to obtain $c(a)=0$. Then P^{hk} is gauge invariant.

Now, from (9) we deduce

$$L-P=H+K \tag{16}$$

where

$$H=d_{\alpha\beta\gamma}\varepsilon^{ijhk}A^{\alpha}_{i,j}A^{\beta}_{h}A^{\gamma}_{k}, \qquad K=K(g_{ij};A_i)$$

From Theorem 1 in López *et al.* (1989) we know that P can be written as

$$P=L_1+H_1 \tag{17}$$

where L_1 is gauge invariant and $H_1=H_1(A^{\alpha}_{i};A^{\alpha}_{i,j})$. But then

$$L=L_1+H+H_1+K=L_1+H_2+K$$

It is easy to see that $E^i_\alpha(H_2)(0;-\frac{1}{2}F_{ij})=0$ (Calvo *et al.*, to appear). The gauge invariance of $E^i_\alpha(L_1)$ follows from the gauge invariance of L_1. Since the same is true for $E^i_\alpha(L)$, then $E^i_\alpha(H_2+K)$ is gauge invariant. Then, since $K^i_\alpha(g_{ij};0)=0$ (Noriega and Schifini, 1985), we have

$$\begin{aligned} E^i_\alpha(H_2+K)&=E^i_\alpha(H_2+K)(g_{ij};0;-\tfrac{1}{2}F^{\alpha}_{ij})\\ &=0+K^i_\alpha(g_{ij};0)=0 \end{aligned}$$

Then L_1 is the Lagrangian we were looking for. We have proved:

Theorem 1. If $L=L(g_{ij};A^{\alpha}_{i};A^{\alpha}_{i,j})$ is a scalar density such that $E^i_\alpha(L)$ is gauge invariant, then there exists a gauge-invariant scalar density L_1 such that $E^i_\alpha(L)=E^i_\alpha(L_1)$.

3. THE UNIQUENESS OF THE YANG–MILLS EQUATIONS

We turn now to the problem of defining the field equations for a Lagrangian $L_1(g_{ij};F^{\alpha}_{ij})$. One could consider the complete set of Yang–Mills equations as

$$B_{\alpha\beta}F^{\beta ij}{}_{;j}=0 \tag{18}$$

$$*F^{\alpha ij}|_j=0 \tag{19}$$

where $*F^{\alpha ij}=g^{-1/2}\varepsilon^{ijhk}F^{\alpha}_{hk}$. For L_1, these could be generalized to

$$E^i_\alpha(L_1)=0 \tag{20}$$

$$*L_1{}^{ij}_{\alpha}|_j=0 \tag{21}$$

Now, in the deduction in López *et al.* (1989) we need L_1^{hk} and $E^i_\alpha(L_1)$ to be gauge-invariant tensorial densities. Here this is true because L_1 is a gauge-invariant scalar density. Then (20) and (21) are (1) and (19) following the proof in López *et al.* (1989). We deduce:

Theorem 2. If $E^i_\alpha(L)$ is gauge invariant for a scalar density $L = L(g_{ij}; A^\alpha_i; A^\alpha_{i,j})$, then the set of equations $E^i_\alpha(L) = 0$ and $*L^{i,j}_\alpha|_j = 0$ implies $B_{\alpha\beta}F^{\beta ij}|_j = 0$ (Yang–Mills equations) and the identity $B_{\alpha\beta}\, {*F^{\beta ij}}|_j = 0$.

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